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RECTANGULAR LAYOUT PROBLEMS WITH WORST-CASE DISTANCE MEASURES.(U)

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RECTANGULAR LAYOUT PROBLEMS WITH WORST-CASE DISTANCE MEASURES

Research Report No. 77-7

by

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August, 1977

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This paper considers two separate, but related problems involving the design of rectangular layouts of m activities. In each of the problems, costs are incurred which are non-decreasing in distance between activities. The distance between two activities is either the worst-cast rectilinear distance, or the worst-case Tchebyshev distance. Minisum and minimax layout problems are then considered and solution techniques are provided.

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Abstract

This paper considers two separate, but related problems involving the design of rectangular layouts of m activities. In each of the problems, costs are incurred which are non-decreasing in distance between activities. The distance between two activities is either the worst-case rectilinear distance, or the worst-case Tchebyshev distance. Minisum and minimax layout problems are then considered and solution techniques are provided.

Introduction and Definitions

This paper considers two separate, but related problems involving the design of rectangular layouts of m activities. In each of the problems, costs are incurred which are nondecreasing in either the rectilinear distance or crane travel time between activities, and it is desired to find rectangular layouts which minimize either a maximum or a total cost. The problems are idealized, and the solutions obtained should thus perhaps best be viewed as design aids.

In what follows we first define the rectangular layouts of interest. Then we state the two problems, and briefly discuss relevant literature. Next we present algorithms which solve the problems, illustrate the use of the algorithms, and identify some insights the algorithms provide. The last part of the paper consists of the analysis needed to justify the algorithms.

In order to facilitate the problem statements and analysis we introduce some definitions. Given m activities, with associated positive numbers $A_1, \ldots, A_m$, we define a <u>layout</u> to be a collection $S = \{S_1, \ldots, S_m\}$ such that S_i, the planar region taken up by activity i, is a compact planar set of area A_i for $i \in M \equiv \{1, \ldots, m\}$, $m \geq 2$, and such that S_i and S_j do not overlap for $i \neq j$. An <u>ordered rectangular layout</u> is a layout S for which there is an ordering of 1, ..., m, say [1], ..., [m] (called the <u>ordering for S</u>) such that $\cup \{S_{[i]} : 1 \leq i \leq j\}$ is a rectangle of area $B_{[j]} \equiv \sum \{A_{[i]} : 1 \leq i \leq j\}$ for $1 \leq j \leq m$. A <u>doubly rectangular</u> (DR) <u>layout</u> is an ordered rectangular layout S for which S_i is the union of a finite collection of rectangles, each of which has the same orientation with respect to the axes, for $i \in M$. Figures 1 - 6 illustrate DR layouts; Fig. 1 gives some indication of how general such layouts can be. We denote the collection of all DR layouts by C_m. A <u>concentric square</u> (CS) <u>layout</u> is a DR layout made up of concentric squares having the ordering [1], ..., [m] from the center outwards. Note that a CS layout is completely defined by its ordering. Fig. 2 illustrates a CS layout.

We shall restrict our analysis to DR layouts, and establish that certain CS layouts are optimal for the problems we consider. In view of the fact that standard construction practices generally dictate that buildings and rooms have rectangular shapes, and in view of the fact that plots of land are usually rectangular, it seems quite reasonable to consider DR layouts.

In order to develop objective functions for the problems of interest we now consider rectilinear and Tchebyshev distances. Given any two points in the plane, $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$, we denote the Tchebyshev distance between X_1 and X_2 by $t(X_1, X_2)$, where

$$t(X_1, X_2) = \max[|x_1 - x_2|, |y_1 - y_2|] .$$

For $i = 1, 2$ let X_i' denote the point obtained by rotating X_i 45 degrees with respect to the axes. We denote the rectilinear distance between $X_1' = (x_1', y_1')$ and $X_2' = (x_2', y_2')$ by $r(X_1', X_2')$, where

$$r(X_1', X_2') = |x_1' - x_2'| + |y_1' - y_2'|.$$

It is known that the two distances are related as follows:

$$r(X_1', X_2') = \sqrt{2}\, t(X_1, X_2). \tag{1}$$

That is, the two distances are the same under a 45 degree rotation and change of scale of $\sqrt{2}$. The relationship (1) is given in [1], and has previously proven useful in the study of rectilinear location problems [2], [3], as well as rectilinear layout problems [5].

Due to the prevailing use of rectilinear aisle (and street) structures, most of the layout problems of interest here involve rectilinear distances. However, for analytical purposes our approach will be to study a simpler

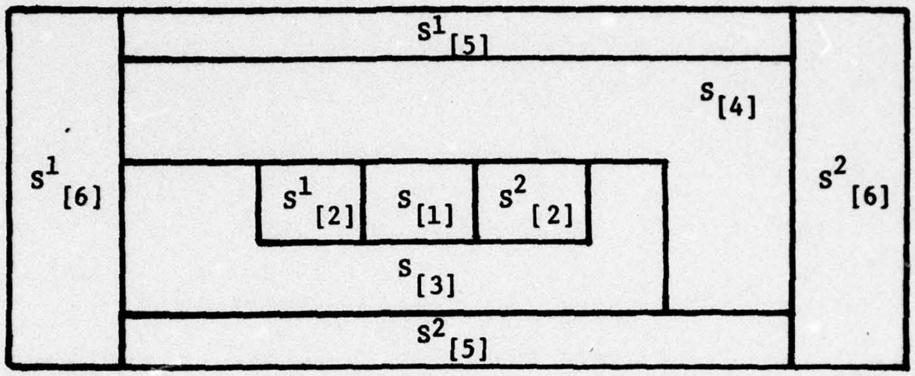


Figure 1. Example of Doubly Rectangular Layout : $S_{[i]} \equiv \bigcup_j S^j_{[i]}$

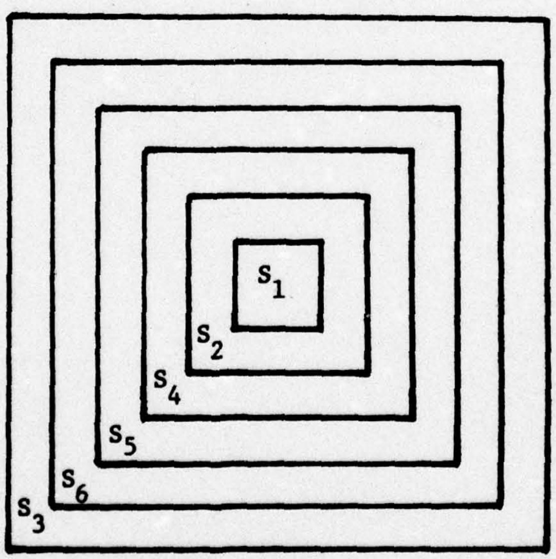


Figure 2. Example Concentric Square Layout for m = 6

layout problem in which the distances are Tchebyshev. Thus, if the original problem is one in which rectilinear distances apply, we rotate any layout under study by 45 degrees relative to the axes and using relation (1), multiply each Tchebyshev distance by $\sqrt{2}$ to obtain the corresponding rectilinear distance. A more detailed discussion of the relation between rectilinear and Tshebyshev distance is given in [5]. We emphasize the fact that all the figures we display are for problems involving Tchebyshev distance, unless noted otherwise. The displayed layouts must be rotated 45 degrees with respect to the axes to obtain the layouts for the equivalent rectilinear distance problem.

We remark here that there is also at least one case where the Tchebyshev distance is of direct interest. Consider the case where a crane can move simultaneously and independently in two orthogonal directions, which are parallel to the two axes. Suppose further that we ignore acceleration and deceleration effects, and choose a measurement scale so that the two movement velocities are the same, v, with v = 1. If the crane is at the point $X_1 = (x_1, y_1)$ and moves to the point $X_2 = (x_2, y_2)$, then the time for the crane to move to any point lying on the line $x = x_2$ is $t_1 \equiv |x_1 - x_2|/v$, while the time for the crane to move to any point lying on the line $y = y_2$ is $t_2 \equiv |y_1 - y_2|/v$. Because the crane is at X_2 if and only if its location is both on the line $x = x_2$ and the line $y = y_2$, the time for it to move to X_2 is the greater of t_1 and t_2, $\max(t_1, t_2)$, which is just the equation (1) when $v = 1$. Thus the layout problems we consider may also be interpreted as problems for which all movement between activities is by crane, and the objective functions which we shall consider are nondecreasing functions of crane travel time.

Given any layout S in C_m we denote by $D_i(S)$ the maximum distance between any point in S_i and any point in $\cup\{S_j : j \in M, j \neq i\}$. That is, if the layout problem involves Tchebyshev distances,

$$D_i(S) = DT_i(S),$$

where

$$DT_i(S) = \max[t(X_1, X_2) : X_1 \in S_i, X_2 \in \cup\{S_j : j \in M, j \neq i\}].$$

If the layout problem involves rectilinear distances, we rotate S by 45 degrees to obtain say, S', and we have

$$D_i(S) = \sqrt{2}\ DT_i(S'),$$

due to (1).

As discussed in [4], $D_i(S)$ is a conservative overestimate, or worst-case measure, of the distance between activity i and the other activities. To illustrate $D_i(S)$, with reference to Fig. 2, if S is a concentric square layout, with order [1], [2], ..., [m], and we define

$$d_{[i]} = (\sqrt{B_{[i]}} + \sqrt{B})/2$$

(where $B \equiv A_1 + \cdots + A_m$), it is easy to verify that when Tchebyshev distances apply, $D_i(S) = d_{[i]}$, $1 \leq i \leq m-1$, $D_m(S) = D_{m-1}(S)$.

Problem Statements and Related Literature

For every activity i in M, we suppose we are given a nondecreasing function f_i which is defined on the nonnegative reals. Given a layout S in C_m we define

$$F_i(S) = f_i\,[D_i(S)],\ i \in M,$$

$$F(S) = \max[F_i(S) : i \in M],$$

$$G(S) = \sum [F_i(S) : i \in M].$$

It should be understood that the functions f_i used in defining F(S) may differ from the ones used in defining G(S): they must, however, have the same properties of being defined on the nonnegative reals and being nondecreasing. In either case, for a layout S in C_m, $F_i(S)$ is a cost incurred for activity i which is nondecreasing in the distance $D_i(S)$. F(S) is then a maximum cost, while G(S) is a total cost. The minimax layout problem is then to find a layout to minimize F(S), while the "minisum" problem is to find a layout to minimize G(S).

A rationale for the objective functions F and G is discussed in [4] in some detail. Briefly, instead of being concerned about distances between every pair of activities, we are concerned about the greatest distance for each activity, and incur costs which are proportional to this greatest distance. A similar rationale is discussed in [5]. We comment that all of the literature relevant to the two problems we consider is discussed in [4] and [5], and so will not be discussed here. The problems considered in [4] are analogs of the problems considered here when layouts must be made on the line, e.g., an aisle. The problems considered in [5] are closely related to the minimax problem considered here when the functions f_i are defined by $f_i(y) = y$, $1 \leq i \leq m$. Thus the contributions of this paper consist of extending the results of [4] to the analogous planar case, and generalizing the objective functions of the planar layout problems considered in [5] for the special case where layouts are doubly rectangular.

So as to make clear the class of layouts over which we shall optimize, when S is a layout in C_m such that each edge in S is parallel to one axis and perpendicular to the other, we call S an <u>aligned</u> doubly rectangular (ADR) layout. A DR layout which is not aligned we call an <u>unaligned doubly rectangular</u> (UDR) layout. Fig. 6 illustrates a UDR layout. Hence a DR layout is either an ADR or a UDR layout. We shall be minimizing over <u>all</u> DR layouts, both ADR and UDR. When an ADR layout is a CS layout we call it an <u>aligned concentric square</u> (ACS) layout.

Given layouts S* and S in C_m for which $D_i(S^*) \leq D_i(S)$ for $i \in M$, we say that S* <u>dominates</u> S. We shall subsequently establish that any layout S in C_m is dominated by an ACS layout S* in C_m. Hence, due to the nondecreasing property of the minimax and minisum objective functions in distance, it will suffice to consider only ACS layouts in order to solve the two problems of interest.

<u>Solving the Minimax Problem</u>

In order to solve the minimax problem we shall assume that activities can be numbered so that $f_i \geq f_{i+1}$, that is, $f_i(y) \geq f_{i+1}(y)$ for all $y \geq 0$, $1 \leq i \leq m-1$. We can always satisfy this assumption when the f_i are increasing linear functions with the property that $f_i(0) = f_j(0)$ for all $i \neq j$, e.g., $f_i(y) = w_i y$, $y \geq 0$, $i \in M$; for this case we number activities so that $w_i \geq w_{i+1}$, $1 \leq i \leq m-1$. Also, we denote by S(p) the ACS layout such that activity p surrounds all other activities and such that, otherwise, activities surround one another in the order of increasing index number from the center square outward. Fig. 2 illustrates S(3) if we suppose [1] = 1, [2] = 2, [3] = 4, [4] = 5, [5] = 6, and [6] = 3.

<u>Minimax Algorithm</u>

(1) Number activities so that $f_1 \geq f_2 \geq \cdots \geq f_m$.

(2) Define $\sigma = m$ if $A_{m-1} \leq A_m$, otherwise define $\sigma = m-1$.

(3) Construct the set $\mathcal{P} \equiv \{p : 1 \leq p \leq m-2, A_m < A_p\} \cup \{\sigma\}$.

(4) Compute $F(S(p))$ for every $p \in \mathcal{P}$.

(5) Any layout $S(p^*)$ for which

$$F(S(p^*)) = \min\{F(S(p)) : p \in \mathcal{P}\}$$

is a minimax layout.

We remark, given layouts $S(p)$ and $S(q)$ with $1 < q < p \leq m$, that $D_i(S(p)) = D_i(S(q))$ for $1 \leq i \leq q-1$. This fact can be useful in computing either of the terms $F(S(p))$ or $F(S(q))$ once the other term is computed.

To illustrate the minimax algorithm, consider an example where Tchebyshev distances apply for which $m = 6$, $f_i(y) = w_i y$, $1 \leq i \leq 6$, $(w_1, w_2, w_3, w_4, w_5, w_6) = (10, 8, 6, 5, 3, 2)$, $(A_1, A_2, A_3, A_4, A_5, A_6) = (33, 2, 4, 5, 8, 10)$. Clearly $f_1 \geq f_2 \geq f_3 \geq f_4 \geq f_5 \geq f_6$. Since $A_5 < A_6$, $\sigma = 6$. $\mathcal{P} = \{1, 6\}$ since A_1 is the only A_i which is greater than A_6. We list the $F_i(S(p))$ for $p = 1, 6$ in Table 1. As can be seen from the table, $p^* = 1$, so $S(1)$ is a minimax layout. Fig. 2 illustrates $S(1)$ on taking $[i] = i+1$ for $1 \leq i \leq 5$ and $[6] = 1$. To solve the analogous minimax problem where rectilinear distances apply, since each f_i is linear, it is easy to show that a 45 degree rotation of $S(1)$ is an optimal layout, but that the optimal objective function value is $\sqrt{2}$ times $F(S(1))$ as given in Table 1.

We emphasize, it is because the functions $f_1, \ldots, f_6$ are linear in this example that the rotated layout for the Tchebyshev problem solves the rectilinear problem. In general, when the functions $f_1, \ldots, f_m$ are nonlinear it is necessary to use the expression $D_i(S(p)) = \sqrt{2}\ DT_i(S(p))$ for computing $F(S(p))$, $p \in \mathcal{P}$.

We now identify some insights obtainable from the minimax algorithm. Consider first the case for which all the A_i are the same: in this case one can verify that an ACS layout with order $1, 2, \ldots, m$ is a minimax layout. Recalling that $f_1 \geq f_2 \geq \cdots \geq f_m$, if we consider f_i to be a measure of the

TABLE 1: Illustration of Minimax Algorithm

i	$F_i(S(1))$	$F_i(S(6))$
1	$10(\sqrt{29} + \sqrt{62})/2$*	$10(\sqrt{33} + \sqrt{62})/2$**
2	$8(\sqrt{2} + \sqrt{62})/2$	$8(\sqrt{35} + \sqrt{62})/2$
3	$6(\sqrt{6} + \sqrt{62})/2$	$6(\sqrt{39} + \sqrt{62})/2$
4	$5(\sqrt{11} + \sqrt{62})/2$	$5(\sqrt{44} + \sqrt{62})/2$
5	$3(\sqrt{19} + \sqrt{62})/2$	$3(\sqrt{52} + \sqrt{62})/2$
6	$2(\sqrt{29} + \sqrt{62})/2$	$2(\sqrt{52} + \sqrt{62})/2$

* $F(S(1)) = F_1(S(1)) \doteq 66.2959$

** $F(S(6)) = F_1(S(6)) \doteq 68.0929$

relative use of activity i, then the minimax layout obtained has the property that the most used activity is in the middle, surrounded by the second most used activity, etc. Alternatively, consider the case where the f_i are identical: in this case, with $A_m = \max(A_i = 1 \leq i \leq m-1)$, one can verify that any ACS layout with activity m surrounding all the other activities is a minimax layout. Motivation for such a layout being minimax can be obtained by recognizing that if activity m did not surround the others, then because activity m has the greatest area the distances to activities surrounding activity m would be increased over and above their value in the minimax layout.

More generally, when neither the f_i nor the A_i are identical, the minimax algorithm demonstrates the fact that we must consider both the relative values of the f_i, and the relative values of the A_i, in order to find a minimax layout.

Solving the Minisum Problem

The minisum problem appears to be a good deal more difficult than the minimax problem. The helpful feature of the minimax problem, in terms of the analysis, is that in seeking a minimax layout from among the ACS layouts most of the effort involves determining which activity should be "outside": the relative positions of other "inside" activities may have little effect on the objective function value. However, for the minisum problem, given an ACS layout S with order [1], ..., [m], every term $F_{[i]}(S)$ has an effect upon the objective function value. Thus the cost depends upon the order, and there are m! possible different orders. Therefore, we need to impose more structure upon the minisum problem in order to solve it.

Given the functions $f_1, \ldots, f_m$ we define the functions $h_1, \ldots, h_m$ by

$$h_i(y) = f_i(y) - f_{i+1}(y), \quad y \geq 0, \quad 1 \leq i \leq m-1, \tag{2A}$$

$$h_m(y) = f_m(y), \quad y \geq 0. \tag{2B}$$

We assume that each of the functions $h_1, \ldots, h_m$ is nondecreasing. When each of the functions $f_1, \ldots, f_m$ is differentiable, with $f_i'(y)$ denoting the derivative of f_i evaluated at y, this nondecreasing assumption is equivalent to

$$f_i'(y) \geq f_{i+1}'(y), \quad y \geq 0, \quad 1 \leq i \leq m-1, \tag{3A}$$

$$f_m'(y) \geq 0. \tag{3B}$$

Minisum Algorithm

Suppose the activities can be numbered so that h_i is nondecreasing for $1 \leq i \leq m$, and so that $A_1 \leq A_2 \leq \cdots \leq A_m$. Construct the ACS layout S* in C_m with ordering 1, 2, ..., m. S* solves the minisum problem.

To illustrate the use of the algorithm, suppose all areas are the same, and that $f_i(y) = w_i y$, $y \geq 0$, $i \in M$. Since $f_i'(y) = w_i$ for $y \geq 0$, if we number the items so that $w_i \geq w_{i+1}$, $1 \leq i \leq m-1$, since $w_m > 0$ the conditions (3), and thus the conditions (2), are satisfied. The resultant ACS layout has the intuitively appealing property that activity 1, which has the largest "weight" (w_1) is placed in the middle, activity 2, with the second largest weight (w_2) surrounds activity 1, etc.

As a second illustration, if we suppose the functions $f_1, \ldots, f_m$ to be identical, then the algorithm constructs an ACS layout for which activity 1, with smallest area (A_1), is in the middle, activity 2, with second smallest area (A_2), surrounds activity 1, etc.

Since we can always number activities so that $A_1 \leq A_2 \leq \cdots \leq A_m$, the only way the conditions the algorithm requires cannot be met is when some function h_i is not nondecreasing: in such a case the minisum problem remains effectively unsolved. However, a result we establish in the analysis (Property 6) may still be useful in reducing the number of ACS layouts which must be considered in order to solve the problem.

Dominance Analysis

In this section we establish that any layout S in C_m is dominated by an ACS layout S* in C_m.

Throughout this section, we assume the distance measure of the original problem is Tchebyshev. This provides no loss of generality since if the distance measure is rectilinear, it is clear that layout S* dominates layout S in rectilinear distance measure if and only if S* dominates S (after each is rotated 45 degrees) in Tchebyshev distance measure.

The dominance proof is by induction, and requires that the dominance results be true for both m = 2 and m = 3 in order to establish the general result. Most of the analysis deals with the case m = 3: once this case is established the general result follows readily.

Throughout this section in order to simplify the notation we assume, given any $S \in C_m$, that activities are numbered so that the ordering for S is 1, 2, ..., m; that is, $[i] = i$ for $i \in M$. Since activities may be numbered arbitrarily our assumption involves no loss of generality.

We use the term distinct layout type to mean a DR layout which cannot be obtained from another DR layout by means of a translation and/or rotation. For any layout $S = \{S_1, S_2\}$ in C_2, either S_2 "surrounds" S_1, (in the sense of Figure 3a) or S_1 and S_2 are adjacent rectangles. Hence we conclude that Fig. 3 illustrates all the distinct layout types for m = 2. Likewise, for m = 3, on letting $T_2 = S_1 \cup S_2$ (which must be a rectangle by the ordering convention) we conclude that all distinct layout types can be obtained using layouts as illustrated in Fig. 4, and the results appear as illustrated in Fig. 5. Thus for m = 3 the analysis may be restricted to the types of layouts shown in Fig. 5. For convenience, we refer to the layout types illustrated

a) $d_i > 0$ for at least two $i \varepsilon \{1, 2, 2, 4\}$

b)

Figure 3. Distinct DR Layouts For m = 2

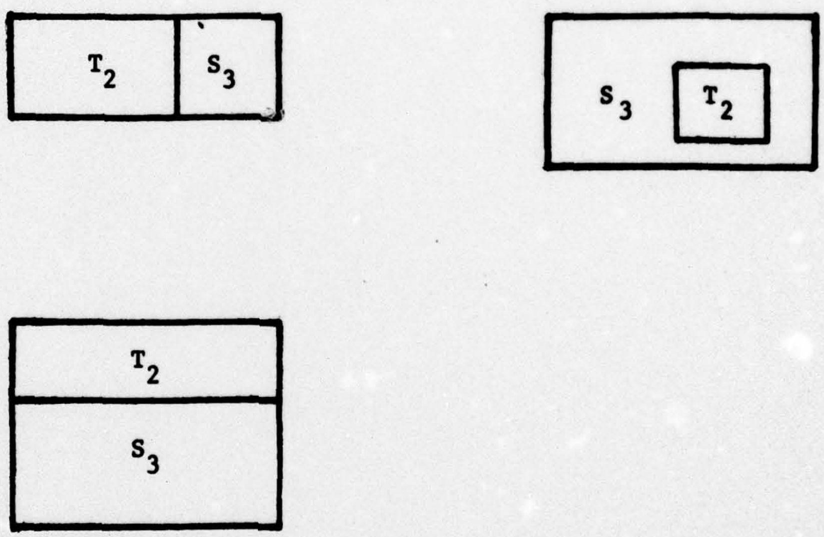


Figure 4. Construction of Distinct DR Layouts For m = 3

in Fig. 5 as Types (a), (b), (c), (d), and (e). The $D_i(S)$ expressions in Fig. 5 are valid when the corresponding layout is aligned; the B and B_i expressions are valid regardless of orientation to the axes.

For an ACS layout S* in C_3 with ordering 1, 2, 3, it is easy to verify that $D_i(S^*) = (\sqrt{B_i} + \sqrt{B})/2$ for i = 1, 2, and $D_3(S^*) = D_2(S^*)$. Thus in order to establish dominance of ACS layouts for m = 3 it is enough to show for any S in C_3, that $D_1(S) \geq (\sqrt{B_1} + \sqrt{B})/2$, $D_2(S) \geq (\sqrt{B_2} + \sqrt{B})/2 \leq D_3(S)$.

The following remark will be useful in the analysis.

Remark. Given positive numbers B_i, B, u, v, and nonnegative numbers a_i, b_i, c_i, d_i, such that

$$(a_i + b_i - u)\ (c_i + d_i - v) = B_i \tag{4}$$

$$uv = B \tag{5}$$

it is true that

$$a_i + b_i + c_i + d_i \geq 2\sqrt{B_i} + 2\sqrt{B}, \tag{6}$$

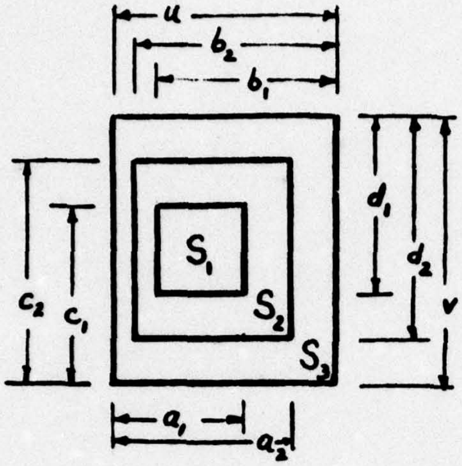
and

$$\max(a_i, b_i, c_i, d_i) \geq (\sqrt{B_i} + \sqrt{B})/2. \tag{7}$$

Proof. By the isoperimetric theorem for rectangles, given any positive numbers x, y, and a for which xy = a, it is true that $x + y \geq 2\sqrt{a}$. Applying this theorem to (4) and (5) we obtain $(a_i + b_i - u) + (c_i + d_i - v) \geq 2\sqrt{B_i}$ and $u + v \geq 2\sqrt{B}$, which give (6) when added. Now if (7) is false, then each of the terms a_i, b_i, c_i, and d_i is strictly less than $(\sqrt{B_i} + \sqrt{B})/2$, and so $a_i + b_i + c_i + d_i < 4[(\sqrt{B_i} + \sqrt{B})/2] = 2\sqrt{B_i} + 2\sqrt{B}$, which contradicts (6). Thus (7) is not false, and the claim is established.

In order to establish dominance of ACS layouts for m = 3, we consider the Type (a) layout of Fig. 5 for both the aligned and unaligned cases. For the remaining layout types of Fig. 5, we consider only the aligned cases.

a)



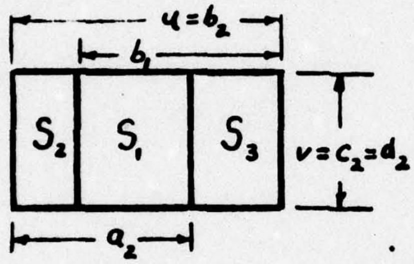
$B = uv$

$B_i = (a_i + b_i - u)(c_i + d_i - v), \quad i=1,2$

$D_i(S) = \max(a_i, b_i, c_i, d_i), \quad i=1,2$

$D_3(S) = D_2(S)$

b)



$B = uv$

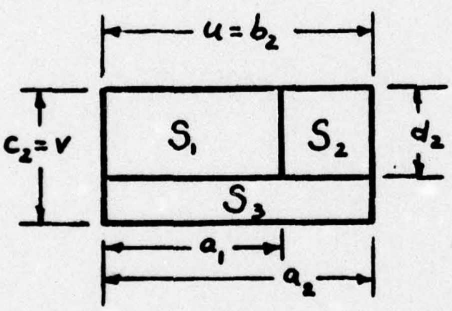
$B_1 = (a_2 + b_1 - u)(c_2 + d_2 - v)$

$B_2 = (a_2 + b_2 - u)(c_2 + d_2 - v)$

$D_1(S) = \max(a_2, b_1, c_2, d_2)$

$D_2(S) = D_3(S) = \max(a_2, b_2, c_2, d_2)$

c)



$B = uv$

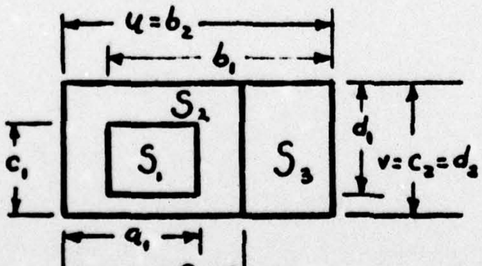
$B_1 = (a_1 + b_2 - u)(c_2 + d_2 - v)$

$B_2 = (a_2 + b_2 - u)(c_2 + d_2 - v)$

$D_1(S) = \max(a_1, b_2, c_2, d_2)$

$D_2(S) = D_3(S) = \max(a_2, b_2, c_2, d_2)$

d)



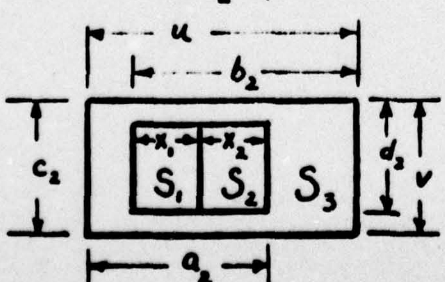
$B = uv$

$B_i = (a_i + b_i - u)(c_i + d_i - v), \quad i=1,2$

$D_i(S) = \max(a_i, b_i, c_i, d_i), \quad i=1,2$

$D_3(S) = D_2(S)$

e)



$B = uv$

$B_1 = (a_2 - x_2 + b_2 - u)(c_2 + d_2 - v)$

$B_2 = (a_2 + b_2 - u)(c_2 + d_2 - v)$

$D_1(S) = \max(a_2 - x_2, b_2, c_2, d_2)$

$D_2(S) = \max(a_2, b_2 - x_1, c_2, d_2)$

$D_3(S) = e_2 = \max(a_2, b_2, c_2, d_2)$

Figure 5 - Distinct DR Layouts for m=3

The analysis for the unaligned cases for types (b) - (e) is basically similar to the unaligned case for the Type (a) layout, but differs in many small details, and is tedious. We refer the reader to [6] for the complete analysis. If the type (a) layout is aligned, then with the dimensions a_i, b_i, c_i, d_i as defined by the figure it is easy to see that $D_i(S) = \max(a_i, b_i, c_i, d_i)$, $i = 1, 2$, and that $D_3(S) = D_2(S)$. Further, the dimensions satisfy (4) and (5) of the Remark, and so we conclude from (7) that S is dominated by an ACS layout. Now suppose the layout in question, say S', is unaligned, and makes an angle with the y axis of θ, as illustrated in Fig. 6. In this figure the perimeter of the innermost rectangle represents the perimeter of either S_1 or S_2, and it can be seen that $D_i(S') = \max(a'_i, b'_i, c'_i, d'_i)$, $i = 1, 2$, (and $D_3(S') = D_2(S')$) where

$$a'_i = a_i \cos\theta + c_i \sin\theta$$

$$b'_i = b_i \cos\theta + d_i \sin\theta$$

$$c'_i = b_i \sin\theta + c_i \cos\theta$$

$$d'_i = a_i \sin\theta + d_i \cos\theta$$

Further, the dimensions of the figure satisfy (4) and (5) of the Remark, and so (7) is true. Now suppose

$$D_i(S') = \max(a'_i, b'_i, c'_i, d'_i) < (\sqrt{B_i} + \sqrt{B})/2 . \tag{8}$$

Then

$$a'_i + b'_i + c'_i + d'_i < 4[(\sqrt{B_i} + \sqrt{B})/2] = 2\sqrt{B_i} + 2\sqrt{B}$$

and, using the equations for $a'_i, \ldots, d'_i$ and simplifying thus gives

$$(a_i + b_i + c_i + d_i)(\sin\theta + \cos\theta) < 2\sqrt{B_i} + 2\sqrt{B} . \tag{9}$$

Now since $0 \le \theta \le 90°$, $\sin^2\theta + \cos^2\theta = 1$ implies $1 \le \sin\theta + \cos\theta$, and so

$$(a_i + b_i + c_i + d_i)1 \le (a_i + b_i + c_i + d_i)(\sin\theta + \cos\theta). \tag{10}$$

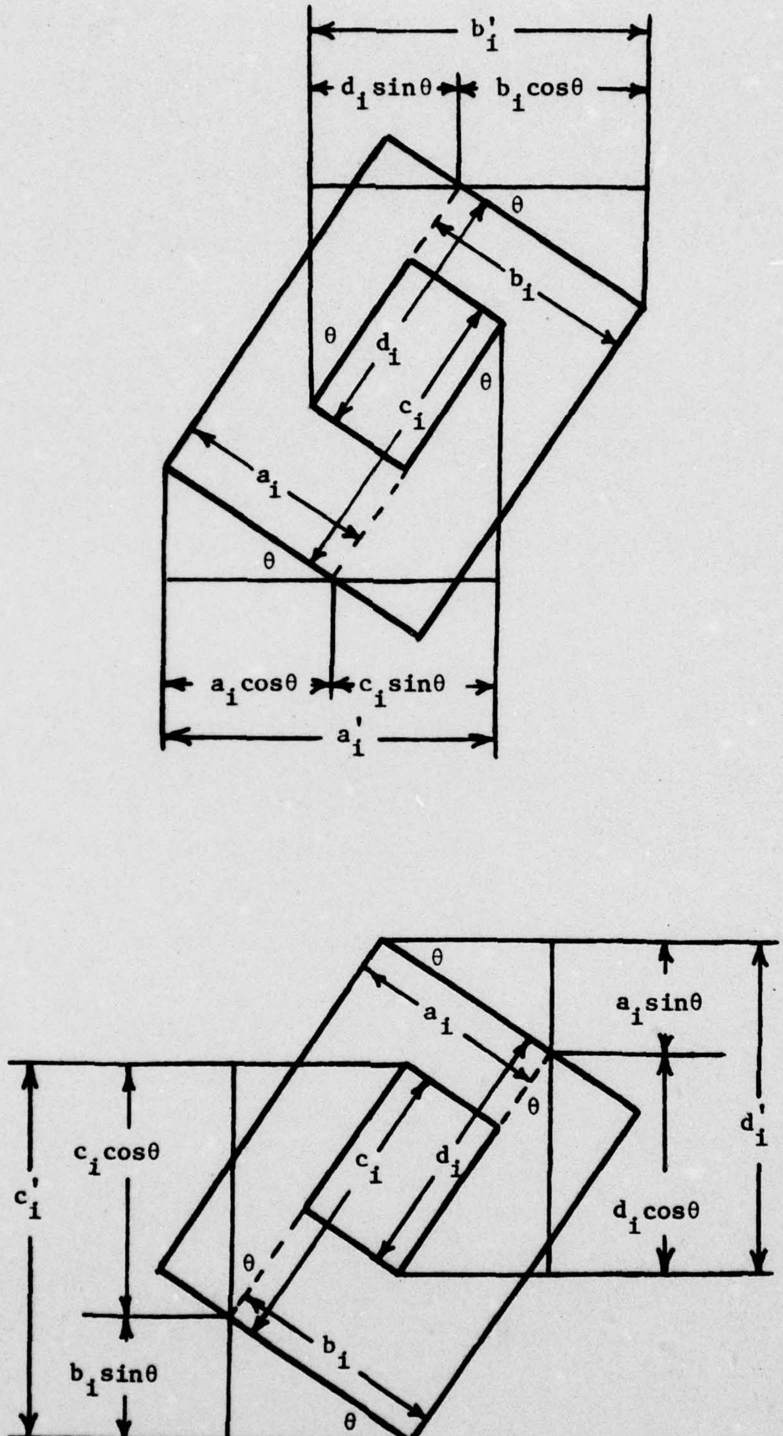


Figure 6. Example of Type a) UDR Layout

Combining (9) and (10) gives $a_i + b_i + c_i + d_i < 2\sqrt{B_i} + 2\sqrt{B}$, which contradicts (7). Thus (8) is impossible, and so, for Type (a) layouts, $D_i(S') \geq (\sqrt{B_i} + \sqrt{B})/2$ for i = 1, 2. Since $D_3(S') = D_2(S')$, the dominance result is established for Type (a) layouts.

Now for $D_1(S)$ for aligned layout Types (b) through (e) one can readily verify using the appropriate dimensions on the figures and the expressions for $D_1(S)$ and B_1, that the analysis is identical to that of $D_1(S)$ for the aligned case of Type (a), and so $D_1(S) \geq (\sqrt{B_1} + \sqrt{B})/2$.

For layout Types (b), (c), and (d), we note that $D_2(S) = \max(a_2, b_2, c_2, d_2) = D_3(S)$, with $(a_2 + b_2 - u)(c_2 + d_2 - v) = B_2$ and $uv = B$. Thus we can employ the Remark to conclude that $D_2(S) = D_3(S) \geq (\sqrt{B_2} + \sqrt{B})/2$.

Finally consider the Type (e) layout. We define $e_2 = \max(a_2, b_2, c_2, d_2)$, and observe $D_3(S) = e_2$. *Further, we may assume* $D_2(S) \geq D_1(S)$ (for otherwise we could renumber activities 1 and 2 and obtain a DR layout satisfying the assumption). Since the dimensions a_2, b_2, c_2, d_2, u and v satisfy the conditions of the Remark, it follows that $e_2 \geq (\sqrt{B_2} + \sqrt{B})/2$. Thus if we can establish $D_2(S) \geq e_2$ our analysis for Type (e) layouts will be complete. Refer to Fig. 5 for the dimensions and the equations for $D_1(S)$ and $D_2(S)$. We consider two cases: (i) $b_2 > c_2$ and (ii) $b_2 \leq c_2$. (i) Given $b_2 > c_2$, suppose $b_2 > \max(a_2, d_2)$. Then $b_2 > \max(a_2, c_2, d_2)$, so $b_2 > \max(a_2, b_2 - x_1, c_2, d_2) = D_2(S)$. But $D_1(S) \geq b_2$ and so $D_1(S) > D_2(S)$, which is a contradiction. Thus $\max(a_2, d_2) \geq b_2$. Since $b_2 > c_2$, $\max(a_2, d_2) > c_2$, so $e_2 = \max(a_2, b_2, c_2, d_2) = \max(a_2, d_2)$. As $D_2(S) \geq \max(a_2, d_2)$, $D_2(S) \geq e_2$. (ii) When $b_2 \leq c_2$, $e_2 = \max(a_2, b_2, c_2, d_2) = \max(a_2, c_2, d_2)$. But clearly $D_2(S) \geq \max(a_2, c_2, d_2)$, and so $D_2(S) \geq e_2$. This completes the analysis for m = 3.

We summarize the foregoing analysis in

Lemma 2. Any layout S in C_3 is dominated by an ACS layout S* in C_3.

We can now establish

Property 1. Given any layout S in C_m,

$$D_i(S) \geq (\sqrt{B_i} + \sqrt{B})/2 \quad , \quad 1 \leq i \leq m-1.$$

$$D_m(S) \geq (\sqrt{B_{m-1}} + \sqrt{B})/2.$$

Proof. We know the claim is true for m = 2 and m = 3, due to Lemmas 1 and 2. Thus assume the claim is true for $m - 1 \geq 3$. It then suffices to show the claim is true for m. Given any layout S in C_m, let us construct the layout $\bar{S}$ as follows. Take $\bar{S}_i = S_i$, $1 \leq i \leq m-2$, $\bar{S}_{m-1} = S_{m-1} \cup S_m$. Likewise define $\bar{A}_i = A_i$, $1 \leq i \leq m-2$, $\bar{A}_{m-1} = A_{m-1} + A_m$. Also let $\bar{B}_i = \bar{A}_1 + \cdots + \bar{A}_i$, $1 \leq i \leq m-2$, $\bar{B} = \bar{A}_1 + \cdots + \bar{A}_{m-1}$. We then note that $\bar{S}$ is in C_{m-1}, $\bar{S}_i$ has area $\bar{A}_i$, $1 \leq i \leq m-1$, $\bar{B}_i = B_i$, $1 \leq i \leq m-2$, and $\bar{B} = B$. Now since $\bar{S} \in C_{m-1}$ the inductive assumption implies

$$D_i(\bar{S}) \geq (\sqrt{\bar{B}_i} + \sqrt{\bar{B}})/2, \quad 1 \leq i \leq m-2.$$

Since $S_i = \bar{S}_i$, $B_i = \bar{B}_i$, for $1 \leq i \leq m-2$, and $B = \bar{B}$, we thus have

$$D_i(S) \geq (\sqrt{B_i} + \sqrt{B})/2, \quad 1 \leq i \leq m-2. \tag{11}$$

Next, for the same given layout S in C_m let us construct the layout $\bar{\bar{S}}$ as follows. Take $\bar{\bar{S}}_1 = S_1 \cup S_2$, $\bar{\bar{S}}_i = S_{i+1}$, for $2 \leq i \leq m-1$. Likewise define $\bar{\bar{A}}_1 = A_1 + A_2$, $\bar{\bar{A}}_i = A_{i+1}$, $2 \leq i \leq m-1$. Also let $\bar{\bar{B}}_i = \bar{\bar{A}}_1 + \cdots + \bar{\bar{A}}_i$, $1 \leq i \leq m-2$, $\bar{\bar{B}} = \bar{\bar{A}}_1 + \cdots + \bar{\bar{A}}_{m-1}$. We then note that $\bar{\bar{S}}$ is in C_{m-1}, $\bar{\bar{S}}_i$ has area $\bar{\bar{A}}_i$ for $1 \leq i \leq m-1$, $\bar{\bar{B}}_i = B_{i+1}$ for $1 \leq i \leq m-1$, and $\bar{\bar{B}} = B$. Now since $\bar{\bar{S}} \in C_{m-1}$ the inductive assumption implies

$$D_i(\bar{\bar{S}}) \geq (\sqrt{\bar{\bar{B}}_i} + \sqrt{\bar{\bar{B}}})/2, \quad 1 \leq i \leq m-2$$

$$D_{m-1}(\bar{\bar{S}}) \geq (\sqrt{\bar{\bar{B}}_{m-2}} + \sqrt{\bar{\bar{B}}})/2.$$

Since $S_{i+1} = \bar{\bar{S}}_i$ for $2 \leq i \leq m-1$, $B_{i+1} = \bar{\bar{B}}_i$ for $2 \leq i \leq m-1$ and $\bar{\bar{B}} = B$, we thus have

$$D_{i+1}(S) \geq (\sqrt{B_{i+1}} + \sqrt{B})/2, \quad 2 \leq i \leq m-2$$

$$D_m(S) \geq (\sqrt{B_{m-1}} + \sqrt{B})/2 \tag{12}$$

so that

$$D_i(S) \geq (\sqrt{B_i} + \sqrt{B})/2, \quad 3 \leq i \leq m-1. \tag{13}$$

Since $m \geq 4$, the conclusion now follows from (11), (12), and (13).

An immediate consequence of Property 1 is

Property 2. In order to find either a minimax or a minisum layout it suffices to consider ACS layouts.

Minimax Problem Analysis

Given an ACS layout S with order [1], ..., [m], if h = [i] we say that activity h is in position i in S. If, in addition, activity j is in position i+1 in S, and we construct the layout $\hat{S}$ from S by interchanging the positions of activities h and j, we say we have constructed $\hat{S}$ from S by making an adjacent interchange of activities h and j. Throughout the analysis for the minimax problem we assume the activities are numbered so that $f_1 \geq f_2 \geq \cdots \geq f_m$.

Lemma 3. Let S be an ACS layout such that for some $j < k$ activities k and j are in positions i and i+1 respectively, for some i, $1 < i \leq m-2$. If the layout $\hat{S}$ is constructed by making an adjacent interchange of activities j and k, then it is true that $F(\hat{S}) \leq F(S)$.

Proof. For S in C_m, activities k and j in positions i and i+1 respectively, and $i+1 \leq m-1$ implies $D_k(S) \leq D_j(S)$. *With* $j < k$, *we have* $f_k \leq f_j$ *and so*

$$F_k(S) = f_k[D_k(S)] \leq f_j[D_j(S)] = F_j(S).$$

Thus

$$F_j(S) = \max[F_k(S), F_j(S)]. \tag{14}$$

Since activity j is in position i in $\hat{S}$ but in position i+1 in S, and i+1 $\leq$ m-1, $D_j(\hat{S}) < D_j(S)$, so f_j nondecreasing implies

$$F_j(\hat{S}) = f_j[D_j(\hat{S})] \leq f_j[D_j(S)] = F_j(S). \tag{15}$$

Since activity k is in position i+1 in S while activity j is in position i+1 in S, $D_k(\hat{S}) = D_j(S)$, so $f_k \leq f_j$ implies

$$F_k(\hat{S}) = F_k[D_k(\hat{S})] \leq F_j[D_j(S)] = F_j(S). \tag{16}$$

From (14), and (15), and (16) we have

$$\max[F_k(\hat{S}), F_j(\hat{S})] \leq \max[F_k(S), F_j(S)]. \tag{17}$$

As all activities other than j and k have the same position in both S and S we have

$$F_i(\hat{S}) = F_i(S) \quad , \quad i \in M, i \notin \{j, k\}. \tag{18}$$

From (17) and (18) we conclude $F(\hat{S}) \leq F(S)$.

Recall that S(p) is the ACS layout such that activity p surrounds all other activities and, otherwise, activities surround one another, from the center outward, in order of increasing index number.

Property 3. If S is an ACS layout with activity p in position m, then $F(S(p)) \leq F(S)$.

Proof. Given the layout S, let us fix the location of the activity p in position m of S. Recalling $f_1 \geq f_2 \geq \cdots \geq f_m$, let us then find that activity having a smallest index among those in positions 1, ..., m-1, and make successive pairwise interchanges as needed until it is in position 1 in a layout, say S{1}. Then let us find that activity in S{1} having a smallest index from among those in positions 2, ..., m-1, and make successive pairwise interchanges as needed until it is in position 2 in a layout, say S{2}. Continuing in this manner we find that the activity in S{i-1} having a smallest index from among those in positions i, ..., m-1, and make successive pairwise interchanges as needed until it is position i in a layout, say S{i}. When i = m-1 we stop.

Due to Lemma 3, $F(S\{m-1\}) \leq \cdots \leq F(S\{1\}) \leq F(S)$. Clearly $S\{m-1\} = S(p)$ and the conclusion follows.

Property 4. For any p, $1 \leq p \leq m-1$, if $A_p \leq A_m$, the $F(S(m)) \leq F(S(p))$.

Proof. In layout $S(p)$, we note that activity m is in position m-1. If we construct the layout $\hat{S}$ from $S(p)$ by interchanging the positions of activities p and m in $S(p)$, it is direct to verify, because $A_p \leq A_m$, that

$$D_m(\hat{S}) \leq D_p(\hat{S}) \leq D_m(S(p)) = D_p(S(p)). \tag{19}$$

Since the positions of all other activities remain unchanged, we have

$$D_i(\hat{S}) \leq D_i(S(p)), \quad i \in M, \quad i \notin \{p, m\}. \tag{20}$$

From (19) and (20) we conclude $F(\hat{S}) \leq F(S(p))$. Since activity m is in position m in $\hat{S}$, Property 3 implies $F(S(m)) \leq F(\hat{S})$. Because $F(\hat{S}) \leq F(S(p))$, we have $F(S(m)) \leq F(S(p))$.

We now have

Property 5. The minimax algorithm constructs a minimax layout.

Proof: By Property 2, to find a minimax layout it suffices to consider only ACS layouts. Property 3 then implies it suffices to consider only the layouts $S(1), \ldots, S(m)$. Now if $A_{m-1} > A_m$ it is easily established that $F(S(m-1)) \leq F(S(m))$. If $A_{m-1} \leq A_m$, Property 4 gives $F(S(m)) \leq F(S(m-1))$. Thus, with reference to step (2) of the algorithm, $F(S(\sigma)) = \min\{F(S(m-1)), F(S(m))\}$. By Property 4, if $A_k \leq A_m$ for $1 \leq k \leq m-2$, then $F(S(m)) \leq F(S(k))$ and so $F(S(\sigma) \leq F(S(m)) \leq F(S(k))$. Thus it suffices to consider only $S(\sigma)$ together with those layouts $S(p)$ in $\{S(1), \ldots, S(m-2)\}$ for which $A_p > A_m$ in order to find a minimax layout.

Minisum Problem Analysis

The following property is the key to the minisum problem analysis.

Property 6. Suppose S is an ACS layout in C_m having activities j and k in positions p and q respectively, where $p < q$. If $A_k \leq A_j$, $f_k - f_j$ is a nondecreasing function, and we construct the layout $\hat{S}$ and S by interchanging the position of activities j and k, then $G(\hat{S}) \leq G(S)$.

Proof. We first consider the case where $q < m$. Denote by I_1, I_2, and I_3 the collections of indices of activities in positions 1 through p-1 in S, p+1 through q-1 in S, and q+1 through m in S. Note $q < m$ implies $I_3 \neq \phi$. S and $\hat{S}$ may be conveniently depicted (with positions from the inside to the outside represented by positions from left to right) as follows:

$$\begin{array}{cccccc} & & \underline{\text{pos. p}} & & \underline{\text{pos. q}} & \\ S: & I_1 & j & I_2 & k & I_3 \\ \hat{S}: & I_1 & k & I_2 & j & I_3 \end{array}$$

Because the positions of activities in I_1 remain unchanged,

$$D_i(\hat{S}) = D_i(S), \quad i \in I_1. \tag{21}$$

Because $A_k \leq A_j$, and activity k is in position p in $\hat{S}$ while activity j is in position p in S,

$$D_k(\hat{S}) \leq D_j(S). \tag{22}$$

Because $A_k \leq A_j$ and activities in I_2 retain the same positions,

$$D_i(\hat{S}) \leq D_i(S), \quad i \in I_2. \tag{23}$$

Because activities j and k are in position q in $\hat{S}$ and S respectively,

$$D_j(\hat{S}) = D_k(S). \tag{24}$$

Since the positions of activities in I_3 remain unchanged,

$$D_i(\hat{S}) = D_i(S), \quad i \in I_3. \tag{25}$$

Since $p < q < m$,

$$D_j(S) < D_k(S), \tag{26}$$

so, since $f_k - f_j$ is a nondecreasing function,

$$f_k[D_j(S)] - f_j[D_j(S)] \leq f_k[D_k(S)] - f_j[D_k(S)].$$

or

$$f_j[D_k(S)] + f_k[D_j(S)] \leq f_j[D_j(S)] + f_k[D_k(S)].$$

Now (24) and f_j nondecreasing implies

$$f_j[D_j(\hat{S})] \leq f_j[D_k(S)],$$

while (22) and f_k nondecreasing implies

$$f_k[D_k(\hat{S})] \leq f_k[D_j(S)].$$

Thus the latter three inequalities imply

$$F_j(\hat{S}) + F_k(\hat{S}) \leq F_j(S) + F_k(S). \tag{27}$$

From (21), (23), and (25) we have

$$\sum \{F_i(\hat{S}) : i \in I_1 \cup I_2 \cup I_3\} \leq \sum \{F_i(S) : i \in I_1 \cup I_2 \cup I_3\} \tag{28}$$

The addition of (27) and (28) gives $G(\hat{S}) \leq G(S)$.

For the remaining case where $q = m$, $I_3 = \phi$. Upon setting $I_3 = \phi$, changing (24) to $D_j(\hat{S}) \leq D_k(S)$, and changing (26) to $D_j(S) \leq D_k(S)$, the analysis goes through exactly as before.

The repeated use of Property 6, together with Property 2, gives

Property 7. If S* is an ACS layout in order 1, 2, ..., m for which $f_i - f_{i+1}$ is a nondecreasing function for $1 \leq i \leq m-1$, and $A_i \leq A_{i+1}$, $1 \leq i \leq m-1$, then S* solves the minisum problem.

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